Chapter 3. The Experimental Basis of Quantum Physics

Notes:

- *Most of the material in this chapter is taken from Thornton and Rex, Chapter 3, and "The Feynman Lectures on Physics, Vol. I" by R. P. Feynman, R. B. Leighton, and M. Sands, Chap. 41 (1963, Addison-Wesley).*

3.1 Discovery of X-rays and the Electron

In the late 1800s cathode ray tubes were commonly used in physics experiments. Although it was not really understood at that time what the exact nature of those "rays" was, they were known to emanate from metal plates placed within evacuated glass cavities (i.e., a vacuum was created within a glass tube) when these plates were subjected to a high electrical potential.

In 1895, the physicist **Wilhelm Röntgen** (1845-1923) discovered that these cathode rays were responsible for the emission of secondary radiation when bombarding the walls of the class tube. Röntgen named this radiation *X-rays*, which he showed to be unaffected by magnetic fields in their trajectory (i.e., they were not electrically charged). He also showed that these rays were very penetrating, as he could even obtain images of bones when X-rays were allowed to pass through the human body (see Figure 1). We now know that X-rays are part of the electromagnetic spectrum located upwards from ultraviolet radiation and downwards of gamma rays in energy (i.e., frequency), in the range of approximately 100 eV and 100 keV (approximately 10^{16} Hz to 10^{19} Hz).

A couple of years later, **J. J. Thompson** (1856-1940) was able to show that cathode rays were negatively charged particles. To do so, he subjected the cathode rays to electric and magnetic fields over a finite spatial range and studied their dynamical behaviour by measuring the displacement they would acquire under these circumstances (see Figure 2). To do so, he considered the Lorentz force acting on the rays, which we assume of charge q

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{3.1}$$

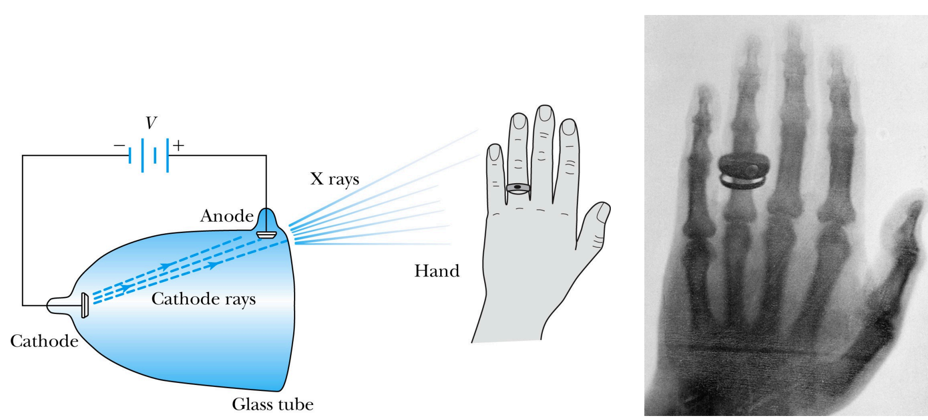

Figure 1 – Schematic of a cathode ray tube and the "X-ray" of a hand.

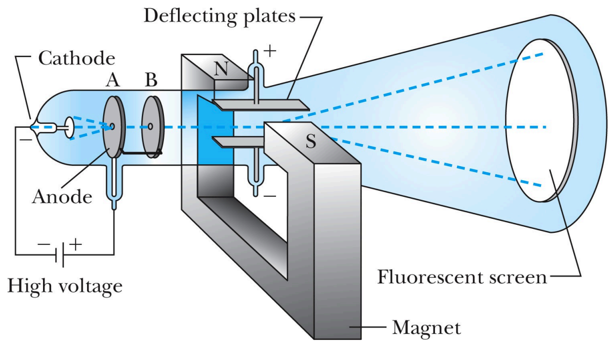

Figure 2 – Schematic representation of the apparatus used by Thompson to study cathode rays.

We will now assume that

$$\begin{aligned}\mathbf{E} &= E\mathbf{e}_y \\ \mathbf{B} &= B\mathbf{e}_z.\end{aligned} \tag{3.2}$$

In the first place, Thompson turned off the magnetic field and according to equation (3.1) got

$$\begin{aligned}F_y &= ma_y \\ &= qE.\end{aligned} \tag{3.3}$$

However, consider that for electrical plates of length ℓ the time of interaction between the charge and the electric field is $t \approx \ell / v_x$ we have for the total angular deflection (i.e., the angle in the trajectory of the ray as it exits the plates)

$$\begin{aligned}\tan(\theta) &= \frac{v_y}{v_x} = \frac{a_y t}{v_x} \\ &\approx \frac{qE}{m}\frac{\ell}{v_x^2}.\end{aligned} \tag{3.4}$$

The only unknown parameter in this equation is the velocity, on which we can also get a handle since it is included of the magnetic part of the Lorentz force. This is done by adjusting the strength of the magnetic field such that the net force on the ray cancels out (i.e., so that $\theta = 0$). That is,

$$\begin{aligned}F_y &= q(E - v_x B) \\ &= 0,\end{aligned} \tag{3.5}$$

or

$$E = v_x B. \tag{3.6}$$

The speed is then determined when the strengths of the electric and magnetic fields are known. Inserting equation (3.6) in (3.4) we get

$$\frac{q}{m} = \frac{E\tan(\theta)}{B^2\ell}, \tag{3.7}$$

and the ratio of the charge to the mass of a cathode ray can be measured experimentally through the deflection angle. In his original experiment Thompson got a value 35% lower than the accepted (absolute) value of 1.76×10^{11} C/kg. The cathode ray particle was eventually named the *electron*.

The charge of the electron was eventually measured to a high accuracy by **Robert A. Millikan** (1868-1953) with an intricate experiment using charged oil drops. The basic idea rests on the fact that oil drops can acquire an electric charge when falling from a metallic nozzle. By subjecting the drops to an electric field that imparted an upward force on them, Millikan was able to balance out the action of gravity such that

$$qE = m_{\mathrm{d}} g \tag{3.8}$$

or

$$q = \frac{m_{\mathrm{d}} g}{E}, \tag{3.9}$$

where m_{d} is the mass of a drop and g is the gravitational acceleration. The mass m_{d} could also be measured by turning off the electric field and measuring the terminal velocity of the drop, while carefully accounting for the drag force acting on it because of air friction. Millikan was then able to show that the charge on an oil drop was always a multiple of some elementary charge quantum, which he determined to be very close to the accepted value of 1.602×10^{-19} C for the electron.

3.2 Line Spectra and the Rydberg Equation

The advent of diffraction grating allowed the development of precise spectrometers, which in turn stimulated the study of spectral lines emitted by different chemical substances. It was realized that the spectra thus recorded acted as some sort of fingerprints and could be used, not only to characterize chemical elements, but also perhaps to learn more about the structure of atoms. A schematic example of a spectrometer with a grating is shown in Figure 3. Radiation from a gas contained in a discharge tube goes through a slit to collimate it into a beam, which then impinges on a diffraction grating. The radiation is then detected on a screen located far away from the grating.

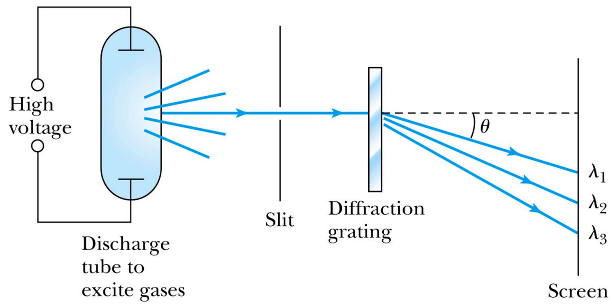

Figure 3 – Schematic example of a spectrometer where radiation from the gas goes through a slit, a diffraction grating, and is detected on a screen located far away.

A diffraction grating is an optical component consisting of a very large number of fine, periodically spaced ruling lines (e.g., slits) of thickness much thinner than the wavelength of the radiation. As the radiation propagates through the grating it diffracts into a pattern that exhibits a series of maxima at angles θ that verify the following relation

$$d\sin(\theta) = n\lambda, \tag{3.10}$$

where d is the distance between adjacent ruling lines, λ the wavelength of radiation, and n an integer called the *order number*.

One of the most salient characteristics of the detected spectra was the fact that radiation happened only at a well-defined set of wavelengths, i.e., there was no continuum radiation. This implied that the changes in energy in the atoms that must accompany the emission of radiation (through the conservation of energy) were quantized in discrete levels. For example, in 1885 **Johann Balmer** (1825-1898) obtained a simple empirical formula to match several lines of the hydrogen spectrum (the so-called *Balmer lines*). This formula was later generalized by **Johannes Rydberg** (1854-1919), in 1890, with the so-called *Rydberg Equation*

$$\frac{1}{\lambda} = R_{\mathrm{H}}\left(\frac{1}{n^2} - \frac{1}{k^2}\right), \tag{3.11}$$

where $R_{\mathrm{H}} = 1.096776 \times 10^7\ \mathrm{m}^{-1}$ is the *Rydberg Constant*, n is an integer that defines a given spectral line series, and $k > n$ is another integer that specifies a line in the series. The series for $n = 1,2,3,4$ and 5 are *Lyman*, *Balmer*, *Paschen*, *Brackett*, and *Pfund Series* of the hydrogen atom, respectively.

3.3 Blackbody Radiation

We briefly discussed blackbodies at the end Chapter 1 when listing the main failures of Classical Physics. We will treat this problem in more details here.

Ideally, a blackbody is one that absorbs all incident radiation, so the only radiation emitted is due to the thermal motion of its charges. Real objects do not quite reach this ideal, but one way to come close is to construct an object with a cavity accessible through a small hole. Any (or, at least, the vast majority of) radiation that enters this hole cannot reflect straight back out. Instead it is absorbed and re-emitted by multiple reflections inside the cavity. The radiation that leaves through the hole is then determined by the thermal motion of charges in the walls of the cavity. Since any accelerating charge radiates electromagnetic radiation and the random motion of these charges depends on the temperature of the object, the amount of radiation emitted (and its spectrum) depends on its temperature T.

3.3.1 Derivation of the Rayleigh-Jeans Law

Lord Rayleigh attempted to explain the blackbody spectrum using all that was then known about physics. His approach was to consider a rectangular cavity (we will use a cubic cavity of side L, for simplicity) and to assume that the cavity contained electromagnetic radiation in the form of all possible normal modes. He further assumed that, in accordance with the equipartition theorem, each of these normal models would contain an average energy kT, i.e., twice $kT/2$ since each corresponds to a standing wave made up of two counter-propagating waves. This allowed him to calculate the energy density within the cavity, and henceforth the amount that would escape through a tiny hole of area A. The result, which we will derive here, is the so-called *Rayleigh-Jeans Law*.

Although the Rayleigh-Jeans Law was unable to match experimentally observed results, its assumptions were reasonable for the time. Indeed, Planck used most of Rayleigh's derivation (with but one additional assumption) when solving the problem. It is then a worthwhile exercise to derive this result. The Rayleigh-Jeans Law is an expression of the total power emitted (per area) by a blackbody as a function of wavelength. Since Rayleigh assumed that the wavelengths were of the normal modes within a cavity, a reasonable starting point is to find an expression for the number of allowed modes at each wavelength.

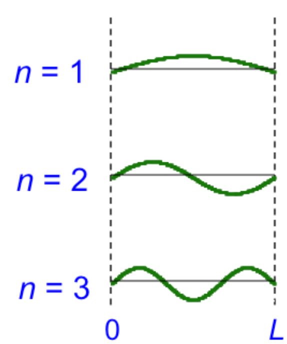

Figure 4 – Example of standing waves between two reflecting surfaces.

As a warm-up, consider waves propagating in a one-dimensional string of length L. If the string is fixed at both ends, so that a wave reaching the end is reflected, the only waves that will be allowed are those for which $n\lambda/2 = L$, with n an integer. In other words, an integral number of *half*-wavelengths must fit on the string. Any other wavelength will interfere destructively with its own reflections, disappearing completely after enough reflections. This is the concept of normal modes. Note that each normal mode exhibits two or more *nodes*, i.e., positions where the deflection of the string remains zero. One node is at each end of the string, and $n-1$ nodes exist between the ends (see Figure 4). The same will be true for electromagnetic standing waves. In this case, the electric field will disappear at the surface of good conducting reflecting surfaces and only modes as defined above will exist. In our case, we generalize, as we need to consider normal modes in three dimensions.

For a blackbody cavity in the form of a cube, the normal modes are standing waves of electromagnetic radiation. Their patterns can also be characterized as fitting within the cavity, but now in three dimensions. We can index these standing waves by the number of half-wavelengths (or, equivalently, the number of nodes) in each of the three directions n_x, n_y, and n_z. If we let these indices represent a point on a three-dimensional plot along the axes $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, we find an infinite number of points forming a cubic grid. A two-dimensional projection is shown in Figure 5. It is important to realize that, although we have thought of our waves as being superpositions of standing waves in the x, y, and z directions, we can form any possible three-dimensional pattern from three waves propagating at some particular angle to the orthogonal axes. The wavelength necessary for this is given by

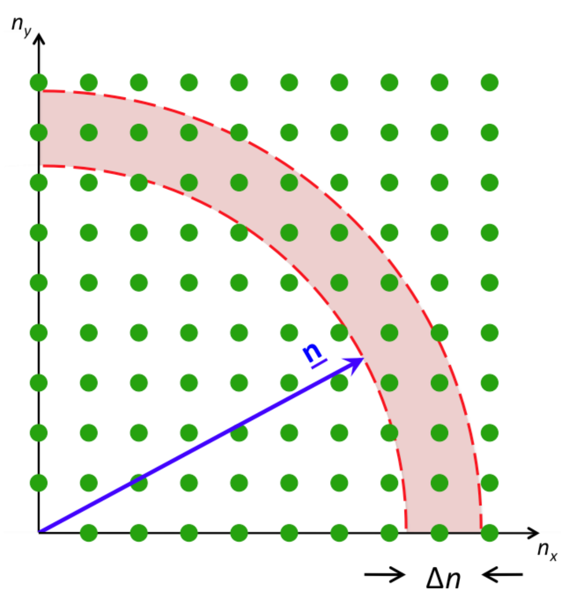

Figure 5 – Two-dimensional projection of the "n-space" mode representation for the possible standing waves in a cube.

$$\lambda_n = \frac{2L}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{2L}{n}. \tag{3.12}$$

The denominator in equation (3.12) can be thought of as the length n of the vector $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y + n_z\mathbf{e}_z$ defined by the point in "n-space" representing the wave in our plot.

The next step is to work out how many modes exist at each wavelength. Since we have conveniently represented our modes as discrete points on a plot along the axes $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, we ask ourselves how many modes exist in a range Δn near any particular value of n. This is shown schematically in the plot as the shaded region in Figure 5. In three dimensions, the shaded region represents one-eighth of a spherical shell (one eighth because the values of each of n_x, n_y, and n_z are restricted to positive integers). The volume of this region is therefore one-eighth the surface area of a sphere of radius n times the thickness Δn of the shell

$$V_{\text{shell}} = \frac{1}{8} \cdot 4\pi n^2 \Delta n = \frac{1}{2}\pi n^2 \Delta n. \tag{3.13}$$

Since there is one mode per unit volume in n-space, this is also the number of modes N in the shell. We will therefore write $N = V_{\text{shell}}$ from now on. We can now use $n = 2L/\lambda$ and $\Delta n = 2L\Delta\lambda/\lambda^2$ to transform equation (3.13) to

$$\begin{aligned} N &= \frac{1}{2}\pi n^2 \Delta n \\ &= \frac{1}{2}\pi \left(\frac{2L}{\lambda}\right)^2 \frac{2L}{\lambda^2}\Delta\lambda \\ &= \frac{4\pi L^3}{\lambda^4}\Delta\lambda \end{aligned} \tag{3.14}$$

for the total number of modes in a range $\Delta\lambda$ around λ. Since each standing wave is made of two counter-propagating waves, it follows that every mode could be thought of as a degree of freedom that contains an average energy kT according to the equipartition theorem (see Section 1.1.4 in Chapter 1). Noting further that there are two linearly independent possibilities for the polarization of each mode, Rayleigh determined that in thermal equilibrium, the total energy of electromagnetic radiation with a wavelength within a range $\Delta\lambda$ around λ should equal

$$
\begin{aligned}
E &= 2 \cdot kT \cdot \frac{4\pi L^3}{\lambda^4}\Delta\lambda \\
&= \frac{8\pi kTV_{\text{cavity}}}{\lambda^4}\Delta\lambda,
\end{aligned}
\tag{3.15}
$$

where in the last step we have replaced L^3 with V_{cavity} the volume of our cubic cavity. The *energy density* per unit wavelength within the cavity is then

$$
u = \frac{8\pi kT}{\lambda^4}. \tag{3.16}
$$

Equation (3.16) is an important result. It predicts that the spectral content of radiation within the cavity will fall off as the fourth power of the wavelength, in agreement with experiments, but increases without bound for small wavelengths, which cannot be right. For example, if you try to integrate this energy density with respect to wavelength to get the total energy at all wavelengths, then you will quickly find that it diverges to infinity. This is the so-called "*ultraviolet catastrophe*."

The final step in deriving the Rayleigh-Jeans Law is to note that the energy escaping through an aperture of area A in the blackbody cavity should be proportional to the energy density in the cavity. We first evaluate the *specific intensity*, which is the energy per unit time, per unit area, per unit solid angle, and unit wavelength

$$
\begin{aligned}
I(\lambda,T) &= \frac{cu}{4\pi} \\
&= \frac{2ckT}{\lambda^4},
\end{aligned}
\tag{3.17}
$$

since in this case the radiation is assumed perfectly isotropic. But what will be measured is the *specific flux* (energy per unit time, per unit area, and unit wavelength)

$$
F(\lambda,T) = \int_0^{\pi/2}\int_0^{2\pi} I(\lambda,T)\cos(\theta)\sin(\theta)\,d\varphi\,d\theta, \tag{3.18}
$$

where θ is the angle between the unit vector pointing outside the box along an axis normal to, and centered on the small aperture A, and φ is the azimuthal angle about that axis. The integration limits are due to the fact that only half of the total radiation is incident on the aperture at angles $0 \le \theta \le \pi/2$ and $0 \le \varphi \le 2\pi$, the other half is moving away from it. Inserting equation (3.17) in (3.18) and integrating we find

$$
F(\lambda,T) = \frac{2\pi ckT}{\lambda^4}. \tag{3.19}
$$

Equation (3.19) is the *Rayleigh-Jeans Law*.

3.3.2 Planck's Solution to the Blackbody Problem

Eventually Planck was able to find an expression for the spectral density that did work. To do this, he had to assume that the energy of radiation in each of the normal modes enumerated by Rayleigh could only take on certain discrete values given by $E_n = nhc/\lambda$, where n is a whole number, and $h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s}$ is a new constant of nature, now known as *Planck's Constant*. One must therefore abandon the equipartition of energy, where it is assumed that each mode contains an average energy of kT. Furthermore, borrowing from the classical Maxwell-Boltzmann distribution he assumed that the probability that a mode of energy E_n is realized is

$$P_n = \alpha e^{-E_n/kT}, \tag{3.20}$$

with α some normalization constant. It follows that the mean energy of a blackbody is

$$\begin{aligned} \langle E \rangle &= \frac{\alpha \sum_{n=0}^{\infty} E_n e^{-E_n/kT}}{\alpha \sum_{n=0}^{\infty} e^{-E_n/kT}} \\ &= \frac{hc}{\lambda} \frac{\sum_{n=0}^{\infty} n\left(e^{-hc/\lambda kT}\right)^n}{\sum_{n=0}^{\infty}\left(e^{-hc/\lambda kT}\right)^n}. \end{aligned} \tag{3.21}$$

If we define $x = e^{-hc/\lambda kT}$, then we can write

$$\begin{aligned} \langle E \rangle (1-x) &= \frac{hc}{\lambda} \sum_{n=0}^{\infty} n\left(x^n - x^{n+1}\right) \Big/ \sum_{n=0}^{\infty} x^n \\ &= \frac{hc}{\lambda} \sum_{n=1}^{\infty} \left[nx^n - (n-1)x^n \right] \Big/ \sum_{n=0}^{\infty} x^n \\ &= \frac{hc}{\lambda} \sum_{n=1}^{\infty} x^n \Big/ \sum_{n=0}^{\infty} x^n \\ &= \frac{hc}{\lambda} \left[1 - \left(1 \Big/ \sum_{n=0}^{\infty} x^n \right) \right]. \end{aligned} \tag{3.22}$$

We furthermore write $A = \sum_{n=0}^{\infty} x^n$ and thus

$$A(1-x) = 1, \tag{3.23}$$

since $\lim_{n\to\infty} x^n = 0$. We therefore have

$$\begin{aligned} \langle E \rangle &= \frac{hc}{\lambda}\frac{x}{1-x} \\ &= \frac{hc}{\lambda}\left(e^{hc/\lambda kT} - 1\right)^{-1}. \end{aligned} \tag{3.24}$$

Finally, we simply have to replace kT in the Rayleigh-Jeans Law of equation (3.19) by equation (3.24) to get

$$F(\lambda,T) = \frac{2\pi c^2 h}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}, \tag{3.25}$$

which is the correct form for the blackbody spectrum. This is **Planck's Radiation Law**. It is straightforward to verify that it reduces to the Rayleigh-Jeans Law as long as $hc/\lambda \ll kT$, and it is then seen to be a long-wavelength approximation to the true law. An example of a blackbody spectrum at $T = 1{,}200\ \text{K}$ together with its Rayleigh-Jeans approximation is shown in Figure 6.

Exercises

1. **Wien's Displacement Law.** Show that the wavelength of maximum specific flux $\lambda_{\max}$ stemming from Planck's Radiation Law is given by

$$\lambda_{\max} T = 2.898 \times 10^{-3}\ \text{m} \cdot \text{K}. \tag{3.26}$$

What is $\lambda_{\max}$ at $T = 1{,}200\ \text{K}$?

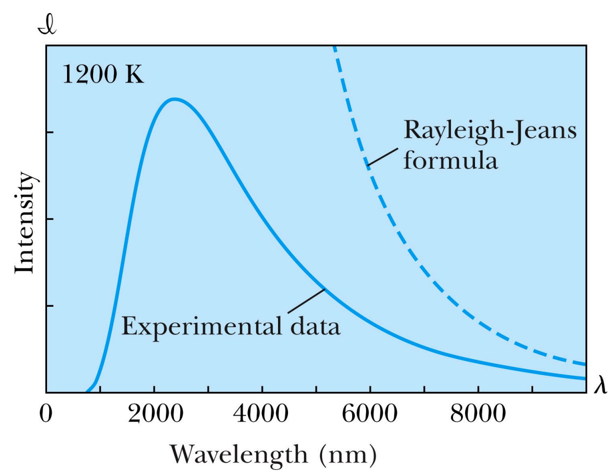

Figure 6 – The blackbody radiation spectrum at $T = 1{,}200\ \text{K}$ (solid curve) together with its Rayleigh-Jeans approximation (broken curve).

Solution.

To find the maximum we must take the derivative of equation (3.25) relative to the wavelength and set it equal to zero. In general, we should also take the second order derivative and verify that it is negative to ensure that we have a maximum in the function (i.e., a second order derivative greater (equal) to zero indicates a minimum (an inflexion point)). However, we know from Figure 6 that we can only have a maximum for a blackbody curve and we will therefore limit ourselves to the first derivative. We thus calculate

$$\begin{aligned}
\frac{dF}{d\lambda} &= 2\pi c^2 h\left[-\frac{5}{\lambda^6\left(e^{hc/\lambda kT}-1\right)}+\frac{1}{\lambda^5}\frac{hc}{\lambda^2 kT}\frac{e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT}-1\right)^2}\right]\\
&= \frac{2\pi c^2 h}{\lambda^6}\frac{1}{e^{hc/\lambda kT}-1}\left(\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT}-1}-5\right)\\
&= 0.
\end{aligned} \tag{3.27}$$

Solving the equation in between parentheses in the second of equations (3.27) we find, by first setting $x = hc/\lambda_{\max} kT$,

$$\frac{xe^x}{e^x-1}-5=0, \tag{3.28}$$

or

$$x = 5\left(1-e^{-x}\right). \tag{3.29}$$

As it turns out, this is a transcendental equation that can only be solved numerically, which when accomplished yields $x \simeq 4.966$. Solving for the product of the wavelength and temperature we have

$$\begin{aligned}
\lambda_{\max} T &= \frac{hc}{xk}\\
&= \frac{6.63\times10^{-34}\left(\mathrm{J\cdot s}\right)\cdot 3.00\times10^{8}\left(\mathrm{m\cdot s^{-1}}\right)}{4.966\cdot 1.38\times10^{-23}\left(\mathrm{J\cdot K^{-1}}\right)}\\
&= 2.898\times10^{-3}\,\mathrm{m\cdot K}.
\end{aligned} \tag{3.30}$$

At a temperature of 1,200 K we have $\lambda_{\max} = 2.42\times10^{-6}\,\mathrm{m}$, the flux thus peaks at near-infrared wavelengths.

2. Integrate Planck's Radiation Law over the whole spectrum to derive the so-called *Stefan-Boltzmann Law*

$$L(T) = \sigma T^4 \tag{3.31}$$

for an ideal blackbody, with

$$\begin{aligned} \sigma &= \frac{2\pi^5 k^4}{15 h^3 c^2} \\ &= 5.6705 \times 10^{-8}\ \mathrm{W}/\left(\mathrm{m}^2 \cdot \mathrm{K}^4\right) \end{aligned} \tag{3.32}$$

the *Stefan-Boltzmann Constant*. What is the radiation power per unit area emanating from a blackbody at a temperature of 1,200 K?

Solution.

We seek to evaluate

$$\begin{aligned} L(T) &= \int_0^\infty F(\lambda, T)\, d\lambda \\ &= 2\pi c^2 h \int_0^\infty \frac{d\lambda}{\lambda^5 \left(e^{hc/\lambda kT} - 1\right)}. \end{aligned} \tag{3.33}$$

We once again make the following change of variable

$$\begin{aligned} x &= \frac{hc}{\lambda kT} \\ dx &= -\frac{hc}{\lambda^2 kT} d\lambda, \end{aligned} \tag{3.34}$$

or

$$d\lambda = -\frac{hc}{x^2 kT} dx, \tag{3.35}$$

which we insert in equation (3.33)

$$\begin{aligned} L(T) &= 2\pi c^2 h \int_\infty^0 \frac{\left(-hc/x^2 kT\right) dx}{\left(hc/xkT\right)^5 \left(e^x - 1\right)} \\ &= 2\pi c^2 h \left(\frac{kT}{hc}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}. \end{aligned} \tag{3.36}$$

The solution for this last integral can be found in a table of finite integrals and equals $\pi^4/15$, which then implies that

$$L(T)=\frac{2\pi^5 k^4}{15h^3c^2}T^4, \tag{3.37}$$

in agreement with equations (3.31) and (3.32).

A blackbody at $T=1{,}200\ \mathrm{K}$ will emit a power per unit area of $1.18\times10^5\ \mathrm{W\cdot m^{-2}}$.

3.4 The Photoelectric Effect

When deriving his law for blackbody radiation, Planck had no explanation to justify his assumption of discrete energies for the degrees of freedom. But he (along with many others) viewed the problem of blackbody radiation to be so important that an explanation had to be found "whatever the cost." He assumed that a better (i.e., "classical") explanation would eventually be found. This, however, never happened. Instead, Einstein realized the significance of this assumption and employed it to solve another outstanding problem in physics: the explanation of the photoelectric effect.

3.4.1 Experimental Results

It was known at the start of the twentieth century that when visible or ultraviolet light was incident on a metal, electrons (i.e., photoelectrons) would be ejected from the surface. The sort of experimental set up used to obtain this result is shown in Figure 7. The photoelectrons emerge from a cathode and are gathered at a collector plate, a voltage difference can be applied between these two plates. The electric current due to these photoelectrons can be measured using an Ampère meter, as shown in the figure. The following experimental facts were known at the time:

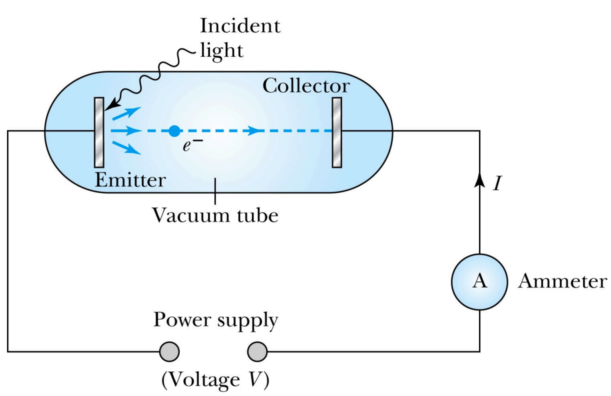

Figure 7 – Experimental set up for the measurement of the photoelectric effect. Photoelectrons are emitted from a cathode when irradiated with visible or ultraviolet light. These electrons are gathered at a collector plate and the current measured.

1. The kinetic energy of the photoelectrons is independent of the intensity of the incident radiation responsible for the ionisation of the cathode.
2. The maximum kinetic energy of the photoelectrons is dependent of the frequency of the radiation.
3. The smaller the ionisation potential (also sometimes called the *work function*) of the cathode material, the lower the threshold frequency at which photoelectrons are ejected.
4. When ionisation occurs the number of photoelectrons measured (with the Ampère meter) is proportional to the light intensity.
5. The photoelectrons are emitted almost instantaneously following the irradiation of the cathode.

Some of these experimental results were contrary to what could be expected from classical physics. More precisely, we would not expect that the kinetic energy of the photoelectrons, or their existence, would depend on the frequency of the radiation but rather on its intensity. That is, the kinetic energy of the photoelectrons should increase with that of the radiation. Similarly, the presence of a frequency threshold for ionisation is completely at odds with classical expectations, which would rather hold that the number of photoelectrons measured be proportional to the intensity of the light *at all levels*. Finally, at very low intensity classical physics would predict that it would take some amount of time before an electron would be sufficiently accelerated to become ionized; this again is contradicted, this time by last fact above.

3.4.2 Einstein's Theory

Einstein's was fully aware of Planck's quantization hypothesis for solving the blackbody radiation problem. Although Planck did not really believe that this hypothesis was based on physical reality, as mentioned above, Einstein postulated that *electromagnetic radiation is quantized* in small bundle of energy. These are the so-called *photons* of energy quantum

$$\begin{aligned} E &= \frac{hc}{\lambda} \\ &= hf \\ &= \hbar\omega, \end{aligned} \tag{3.38}$$

where we recognise Planck's hypothesis in the first of these equations, and with a new form of Planck's constant $\hbar \equiv h/2\pi = 1.0546 \times 10^{-34}\ \mathrm{J \cdot s}$, and $f = c/\lambda$ the frequency of the radiation (of course, $\omega \equiv 2\pi f$).

Einstein then suggested in his landmark 1905 papers that a photon absorbed by the cathode would transfer *all* of its energy to one electron, and that if this energy exceeded the ionisation energy of the electron then it would be ejected from the metal plate. Furthermore, any excess energy beyond that needed for ionisation is converted to kinetic energy for the photoelectron. Since some energy may be needed for the electron to make its way out of the cathode (e.g., from collisions with other electrons populating the metal), it follows that this excess energy acquired from the absorption of the ionizing photon corresponds to the maximum kinetic energy that a photoelectron may have. Einstein therefore wrote down the following equation

$$hf = \phi + \frac{1}{2} m v_{\max}^2, \tag{3.39}$$

where ϕ is the electron work function and $v_{\max}$ its maximum speed after ejection. Einstein's theory is in perfect agreement with the aforementioned experimental facts.

It is important to realize that Einstein's proposal was most daring, since the wavelike nature of electromagnetic radiation had been firmly established for quite some time. He therefore asserted the counterintuitive notion that, apart from its wavelike nature, *light should also be expected to exhibit particle-like characteristics*. It is interesting to note that it is for this work that Einstein was awarded the Nobel Prize in Physics in 1921, not for his relativity theories (special in 1905 and general in 1915).

Exercises

3. (Ch. 3, Prob. 36 in Thornton and Rex.) A 2.0-mW green laser ($\lambda = 532$ nm) shines on a cesium photocathode ($\phi = 1.95$ eV). Assume an efficiency of 10^{-5} for producing photoelectrons (that is, one photoelectron is produced for every 10^5 incident photons) and determine the photoelectric current.

Solution.

The energy of a photon is

$$\begin{aligned} \frac{hc}{\lambda} &= \frac{1240\,\mathrm{eV \cdot nm}}{532\,\mathrm{nm}} \\ &= 2.33\ \mathrm{eV} > \phi \end{aligned} \tag{3.40}$$

such that it possesses enough energy to overcome the electron work function. The photoelectric current can then be calculated with

$$\begin{aligned} I_{\mathrm{e}} &= \underbrace{\left(2 \times 10^{-3}\ \mathrm{J/s}\right)}_{\text{laser power}} \underbrace{\left(\frac{1\ \mathrm{eV}}{1.60 \times 10^{-19}\ \mathrm{J}}\right)}_{\text{J to eV conversion}} \\ &\quad \times \underbrace{\left(\frac{1\ \text{photon}}{2.33\mathrm{eV}}\right)\left(\frac{10^{-5}\ \text{electrons}}{1\ \text{photon}}\right)}_{\text{number of photons for one photoelectron}} \underbrace{\left(\frac{1.60 \times 10^{-19}\mathrm{C}}{\text{electron}}\right)}_{\text{charge per electron}} \\ &= 8.58\ \mathrm{nA} \end{aligned} \tag{3.41}$$

4. (Ch. 3, Prob. 57 in Thornton and Rex.) A typical person can detect light with a minimum intensity of $4.0\times10^{-11}\ \mathrm{W/m^2}$. For light of this intensity and $\lambda = 550\ \mathrm{nm}$, how many photons enter the eye each second if the pupil is open wide with a diameter of $9.0\ \mathrm{mm}$?

Solution.

The radiant power received by the eye is

$$\begin{aligned} P &= 4.0\times10^{-11}\,\mathrm{W/m^2}\cdot\pi\left(4.5\times10^{-3}\,\mathrm{m}\right)^2 \\ &= 2.5\times10^{-15}\,\mathrm{J/s}, \end{aligned} \tag{3.42}$$

and the energy per photon is

$$\begin{aligned} E &= \frac{hc}{\lambda} \\ &= \frac{\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)\left(3.00\times10^{8}\ \mathrm{m/s}\right)}{5.50\times10^{-7}\ \mathrm{m}} \\ &= 3.6\times10^{-19}\ \mathrm{J}. \end{aligned} \tag{3.43}$$

The number of photons entering the eye per second is therefore

$$\begin{aligned} n_{\mathrm{p}} &= \frac{P}{E} \\ &= 2.5\times10^{-15}\ \mathrm{J/s}\cdot\frac{1\ \text{photon}}{3.6\times10^{-19}\ \mathrm{J}} \\ &= 6900\ \text{photons}. \end{aligned} \tag{3.44}$$

3.5 The Compton Effect

Another experimental fact that could not be explained by classical physics is that studied by **Arthur Compton** (1892-1962) in 1923 concerning the scattering of photons by electrons bounded to atoms. According to classical physics, an atomically bound electron can, to a fair level of approximation, be modeled as being harmonically tied to the nucleus (i.e., as in “attached with a spring”). The scattering of electromagnetic radiation (i.e., of photons) would consist of the electron oscillating harmonically at the frequency of the radiation (through coupling due to the Lorentz force $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$) and re-radiating (i.e., scattering) at that same frequency because of the acceleration brought about by its oscillations. This process is called *Thomson scattering*. But Compton observed that there was a longer-wavelength component (the *modified wave*), in addition to expected component at the incident wavelength (the *unmodified wave*), in the scattered radiation. Compton was only able to explain this phenomenon using Einstein’s photon concept.

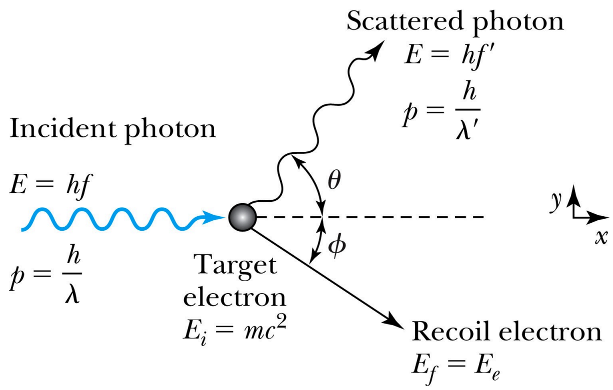

Figure 8 – Photon-electron scattering process studied by Compton, in a reference frame where the electron is initially at rest.

The scattering process studied by Compton is shown in Figure 8. We assume that the scattering takes place in the xy-plane with the incident photon moving along the positive x-axis, the electron initially at rest, and the scattered photon and recoil electron making angles θ and ϕ (as shown in the figure) with the x-axis, respectively.

We know from Special Relativity that the momentum of a photon (or any zero-mass particle) is given by $p = E/c$, which becomes when using the photon concept

$$\begin{aligned} p_{\mathrm{p1}} &= \frac{hf}{c} = \frac{h}{\lambda} \\ p_{\mathrm{p2}} &= \frac{h}{\lambda'}, \end{aligned} \tag{3.45}$$

for the initial and final states, respectively. For the electron, its energy is in general related to the momentum through

$$E_{\mathrm{e}}^2 = \left(mc^2\right)^2 + \left(p_{\mathrm{e}}c\right)^2, \tag{3.46}$$

and we have its initial and final states

$$\begin{aligned} E_{\mathrm{e1}} &= mc^2 \\ E_{\mathrm{e2}}^2 &= \left(mc^2\right)^2 + \left(p_{\mathrm{e2}}c\right)^2. \end{aligned} \tag{3.47}$$

We solve the problem by applying the conservations of energy and linear momentum (along the x- and y-axes), which gives the following relations

$$\begin{aligned}\frac{hc}{\lambda}+mc^2 &= \frac{hc}{\lambda'}+E_{\mathrm{e}2}\\ \frac{h}{\lambda} &= \frac{h}{\lambda'}\cos(\theta)+p_{\mathrm{e}2}\cos(\phi)\\ 0 &= \frac{h}{\lambda'}\sin(\theta)-p_{\mathrm{e}2}\sin(\phi).\end{aligned}\tag{3.48}$$

Our goal is to determine the change in wavelength of the scattered photon $\Delta\lambda=\lambda'-\lambda$. The last two of equations (3.48) can be combined after isolating the terms containing the electron momentum and squaring the resulting equations

$$p_{\mathrm{e}2}^2=\left(\frac{h}{\lambda}\right)^2+\left(\frac{h}{\lambda'}\right)^2-2\frac{h^2}{\lambda\lambda'}\cos(\theta).\tag{3.49}$$

We can now replace $E_{\mathrm{e}2}$ and $p_{\mathrm{e}2}$ in the second of equations (3.47) by the first of (3.48) and (3.49), which gives us

$$h^2c^2\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)^2+2mc^3h\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)+\left(mc^2\right)^2=\left(mc^2\right)^2+\frac{h^2c^2}{\lambda^2}+\frac{h^2c^2}{\lambda'^2}-2\frac{h^2c^2}{\lambda\lambda'}\cos(\theta),\tag{3.50}$$

or

$$mc\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\frac{h}{\lambda\lambda'}\left[1-\cos(\theta)\right]\tag{3.51}$$

We finally obtain the desired result

$$\begin{aligned}\Delta\lambda &= \frac{h}{mc}\left[1-\cos(\theta)\right]\\ &= \lambda_{\mathrm{C}}\left[1-\cos(\theta)\right],\end{aligned}\tag{3.52}$$

where $\lambda_{\mathrm{C}}\equiv h/mc=2.426\times10^{-3}\,\mathrm{nm}$ is the *Compton wavelength*. It is therefore found that no change in wavelength occurs for strictly forward scattering (i.e., $\theta=0$), while it is maximum for the completely backward scattering case (i.e., $\theta=\pi$). In view of the small value of λ_{C}, it follows that the effect will only be important and measurable for sufficiently short wavelengths, such as with X- and gamma rays. At these wavelengths a photon is sufficiently energetic that a loosely bound electron (to an atom) essentially appears as "free" and the photon scatters elastically in the manner calculated above. In the case where a photon is incident on a tightly bound electron the photon effectively scatters off the whole atom to which the electron is bound. The mass that enters in the Compton wavelength is that of the atom, and the effect becomes negligible (i.e., $\Delta\lambda\simeq0$). It follows that experimental data should show photons scattered both at *modified* and

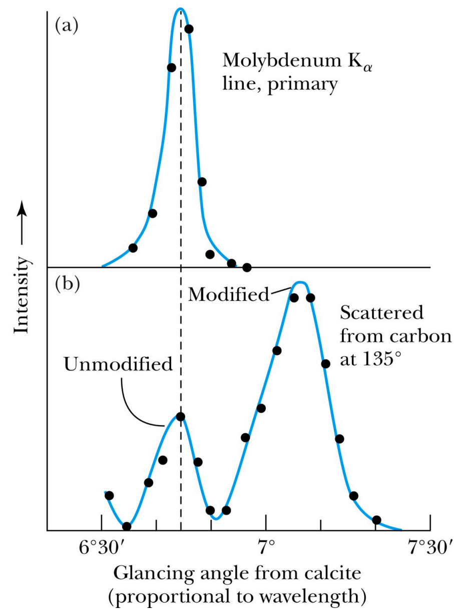

Figure 9 – Experimental data obtained by Arthur Compton, which show the unmodified and modified waves.

unmodified wavelengths. The results Compton obtained, presented in Figure 9, show just that.

Exercises

5. (Ch. 3, Prob. 46 in Thornton and Rex.) Calculate the maximum $\Delta\lambda/\lambda$ of Compton scattering for blue light ($\lambda = 480$ nm). Could this be easily observed?

Solution.

From equation (3.52) the maximum wavelength change happens for $\theta = \pi$, which yields

$$\frac{\Delta\lambda}{\lambda} = \frac{2\lambda_C}{\lambda} = \frac{2 \cdot 2.426 \times 10^{-3}\,\text{nm}}{480\ \text{nm}} = 1.01 \times 10^{-5}. \tag{3.53}$$

This is not an easily observable effect ($\Delta\lambda = 4.853 \times 10^{-3}\,\text{nm}$).

6. (Ch. 3, Prob. 47 in Thornton and Rex.) An X-ray photon having 40 keV scatters from a free electron at rest. What is the maximum kinetic energy that the electron can obtain?

Solution.

Again the maximum effect and gain in energy for the electron happens for $\theta = \pi$. Because of conservation energy, the kinetic energy of the electron is

$$K_{\mathrm{e}}=\frac{hc}{\lambda}\left(1-\frac{1}{1+\Delta\lambda/\lambda}\right). \tag{3.54}$$

The wavelength of the photon is

$$\lambda=\frac{hc}{E}=\frac{1240\ \mathrm{eV}\cdot\mathrm{nm}}{40000\ \mathrm{eV}}=0.0310\ \mathrm{nm} \tag{3.55}$$

while the proportional change in wavelength given by equation (3.53) is $\Delta\lambda/\lambda=0.157$ and therefore $K_{\mathrm{e}}=5.43\ \mathrm{keV}$ from equation (3.54).

7. (Ch. 3, Prob. 49 in Thornton and Rex.) Is it possible to have a scattering similar to Compton scattering from a proton in H_2 gas? What would the Compton wavelength for a proton? What energy photon would have this wavelength?

Solution.

The proton mass ($1.67\times10^{-27}\,\mathrm{kg}$) in energy units is $m_{\mathrm{p}}c^2=938.3\,\mathrm{MeV}$, which yields a Compton wavelength of

$$\lambda_{\mathrm{C}}=\frac{h}{m_{\mathrm{p}}c}=\frac{hc}{m_{\mathrm{p}}c^2}=\frac{1240\ \mathrm{eV}\cdot\mathrm{nm}}{938.3\ \mathrm{MeV}}=1.32\times10^{-6}\ \mathrm{nm}. \tag{3.56}$$

Inverting this equation we find that $hc/\lambda_{\mathrm{C}}=m_{\mathrm{p}}c^2$, i.e., the corresponding photon energy is that of the proton (938.3 MeV). In principle this could be observed, but the energy requirements are high.